SOLOW'S ORIGINAL MODEL VERSUS TEXTBOOK MODEL

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Abstract

The textbook explanation of the Solow model assumes implicitly that there are stationary, rather than steady-state, conditions. This study shows that the stationary-state conditions assumption is not compatible with the neoclassical postulate. The Solow model explains long-term equilibrium at full employment. In the short-term economy is out-of-equilibrium. It is shown that if it is assumed stationary state conditions, there is no growth, saving propensity cannot change exogenously and transition from full employment to out-of-equilibrium conditions, and consequently from out-of-equilibrium to full employment conditions cannot be explained. Explaining the Solow model starting from stationary-state conditions prevents to understand the critical difference between classical and neoclassical economics.

Keywords: Solow model, neoclassical economics, steady state, stationary state.

Introduction

The Solow model is commonly explained by assuming that the growth rate of the labor force is equal to zero (see for example; Mankiw, 2003)[1]. However, once this simplifying assumption has been made, one cannot explain transition from full employment to out-of-equilibrium conditions. Since transition from full employment to out-of-equilibrium conditions cannot be explained, transition from out-of-equilibrium to full employment conditions cannot be explained either. As a result, the neoclassical adaptation mechanism, and the relation between the natural and the warranted rate of growth cannot be explained either.

If the economy is out-of-equilibrium, there is disequilibrium i) in the factor markets, and ii) between the natural and the warranted rates of growth. From the neoclassical perspective, this implies that equilibrium can only be associated with full-employment of the available labor force. The neoclassical adaptation mechanism depends on the factor markets and, due to the supply and demand relations in labor and capital market, the economy moves toward to the steady-state. However, if it is assumed that the growth rate of labor force is equal to zero, in other words if it is assumed that there are *stationary*, rather than *steady-state* conditions, one cannot explain transition i) **from full employment to out-of-equilibrium**[2] conditions, ii) so, from out-of-equilibrium to full employment conditions *since growth rate is equal to zero and saving propensity cannot change in stationary-state but growth rate is*

[1] We need to emphasize that this is a simplifying assumption which is made initially. After this simplification, labor force growth and technological progress are included in the analysis.

[2] One of the crucial assumptions in the Solow model is *full employment*. Indeed, according to Solow 'net investment' and 'total employment' equations are defined assuming that full employment is perpetually maintained (Solow 1956, 66). Solow concludes of his model by indicating some of the obstacles to full employment: "Everything above is the neoclassical side of the coin. Most especially it is full employment economics - in the dual aspect of equilibrium condition and frictionless, competitive, causal system. All the difficulties and rigidities which go into modern Keynesian income analysis have been shunted aside." (Solow 1956, 91) However, in the short-term, in the absence of technical change, due to changes in either the saving propensity, the growth rate of the labor force or the depreciation rate, out-of-equilibrium conditions emerge. In other words, in the absence of technical change, if the saving propensity, the growth rate of the labor force and the depreciation rate gives a value which is different from steady-state value, then out-of-equilibrium conditions emerge. After out-of-equilibrium conditions, transition to equilibrium begins to occur.

positive and saving propensity can change in steady-state. [3] As a result, the mechanism in the factor markets, and the relation between the natural and the warranted rates of growth cannot be explained either. **In other words, if there is no deviation from equilibrium, then, moving toward equilibrium cannot be occurred.**

Herein, it is attempted to clarify the problem and it is argued that when explaining the Solow model, a positive value needs to be assumed for the growth of the labor force.

In the following section, the textbook explanation of the stability of the steady-state in the fundamental Solow model, and then Solow's original explanation are presented. We also provide a numerical example. Finally, conclusions are presented.

The Textbook and Original Explanation of the Stability in the Solow Model

In brief, the textbook explanation of the stability of the Solow model is as follows. Accumulation of capital is given by

$$\Delta k(t) = sy(t) - \delta k(t) \tag{1}$$

where $k(t)$ is the per capita capital stock, s is the saving rate, $y(t)$ is the per capita output and δ is the depreciation rate. Given that stationary-state conditions growth rate of labor and capital stock equal to zero and we rewrite (1) as follows:

$$sy(t) = \delta k(t) \tag{2}$$

Assume that due to an exogenously rise in saving rate, the economy has a capital-to-labor ratio below the stationary-state value. Thus, the desired saving is greater than depreciation:

$$s\,y(t) > \delta\,k(t) \tag{3}$$

Then excess saving is invested, capital accumulates and the economy moves to the stationary-state value of $k(t)$. This process continues until the stationary-state value of $k(t)$ is reached, at a diminishing rate of growth thanks to the diminishing returns to capital. At the stationary-state value of $k(t)$, growth rate of labor and capital stock equal to zero and $k(t)$ remains constant.

The explanation above shows the transition from out-of-equilibrium to equilibrium conditions. Now we explain the stability of the steady-state based on Solow (1956). We try to show that out-of-equilibrium conditions can be occurred when economy is at steady-state rather than stationary–state, initially. Thus, we try to show that out-of equilibrium conditions cannot be occurred at stationary state, and if out-of equilibrium conditions cannot be occurred, of course, reverse movement (moving toward equilibrium) and the postulate of the neoclassical theory cannot be explained either.

Stationary state, steady state and saving rate

[3] It should be emphasized that out-of-equilibrium conditions can emerge not only due to a change in saving propensity but also a change in growth rate of the labor force and depreciation rate. However, in contrast to the endogeneity of the natural rate of growth literature (see; Thirlwall, 2002), the Solow model assumes growth rate of the labor force exogenous. It is also not expected that there can be a remarkable change in the depreciation rate. Thus, it should be focused on the saving propensity as a factor that causes the economy to move away from equilibrium since it can change due to the decisions of the agents.

The Solow model is based on steady state, not stationary state. "In the long-period equilibrium of the stationary state investment must, of course, be nil" (Pigou 1943, 345). Note that, if the growth rate of the labor force is assumed to be zero, the growth rate of the capital stock will also equal to zero; i.e. net investment is zero at stationary state. More importantly, according to Solow model, saving and net investment is a function of yield of capital stock. **If net investment is zero and capital does not change (if it is assumed to be stationary state), then yield of capital will not change either; i.e. saving propensity cannot change exogenously.** Indeed Solow (1956, 87-88) points out the following: "As long as real income was positive, positive net capital formation must result. This rules out the possibility of a Ricardo-Mill stationary state, and suggests the experiment of letting the rate of saving depend on the yield of capital. If savings can fall to zero when income is positive, it becomes possible for net investment to cease and for the capital stock, at least, to become stationary. There will still be growth of the labor force, however; it would take us too far a field to go wholly classical with a theory of population growth and a fixed supply of land." (Solow 1956, 87-88)

Therefore, Solow (1956) obviously disentangles his steady state analysis from stationary state; i.e. neoclassical economics from classical economics.

Now we need to emphasize the main neoclassical postulate:

There is a stable relation between the natural and the warranted rates of growth. The warranted rate of growth will adapt itself in line with the natural growth rate. This mainly occurs thanks to the assumptions of diminishing returns and substitutability of factors, and based on an adaptation mechanism in factor markets.

Let us to explain the problem between this postulate and stationary state.

Solow's fundamental equation is as follows (Solow 1956, 69):

$$\dot{r}(t) = r(t)\frac{sF[K(t), L(t)]}{K(t)} - n\frac{K(t)}{L(t)} \tag{4}$$

where $\dot{r}(t)$ represents change in the capital-to-labor ratio, $K(t)$ is the capital stock, $L(t)$ is labor and n is the growth rate of labor. Note that labor grows at a given and constant rate. If $[r(t)sF[K(t), L(t)]/K(t)] > [nK(t)/L(t)]$ holds, "capital and output will grow at a faster rate than the labor force until the equilibrium ratio is approached" (Solow 1956, 71). Thus "when production takes place under the usual neoclassical conditions of variable proportions and constant returns to scale, no simple opposition between natural and warranted rates of growth is possible." (Solow 1956, 73) **Thus, if out-of equilibrium conditions cannot be occurred, then, moving toward equilibrium, so, the stable relation between the natural and the warranted rates of growth cannot be explained either.**

Note that the debate is whether there can be a persistent divergence between the natural and warranted growth rates and the Solow model gives an answer to the effect that the warranted growth will adapt itself in line with the natural growth rate. This debate is important in the context of the transition from the Harrod and the Domar models to the Solow model. Harrod defines warranted rate of growth as follows: "The warranted rate of growth is taken to be that rate of growth which, if it occurs, will leave all parties satisfied that they have produced neither more nor less than the right amount." (Harrod 1939, 16) The natural rate of growth is "the maximum rate of growth allowed by the increase of population, accumulation of capital, technological improvement and the work/leisure preference schedule, supposing that there is always full employment in some sense." (Harrod 1939, 30) Warranted and natural rates of growth are actually growth rates of capital and labor, respectively (Thirlwall 2002, 15). The relation between them is an analytical tool especially for Post-Keynesian theories (see; Thirlwall 2001). As mentioned above, the relation between warranted and natural rates of growth is in the scope of the Solow's article and Solow only shows that the warranted growth will adapt itself in line with the natural growth rate. **However, this adaptation between warranted and natural rates of growth cannot be occurred since out-of equilibrium cannot be occurred either.**

Besides, Solow, in the introduction of his article, emphasizes substitution of factors during adaptation between warranted rate of growth and natural rate of growth. "In Harrod's terms the critical question of balance boils down to a comparison between the natural rate of growth which depends, in the absence of technological change, on the increase of the labor force, and the warranted rate of growth which depends on the saving and investing habits of households and firms. But this fundamental opposition of warranted and natural rates turns out in the end to flow from the crucial assumption that production takes place under conditions of fixed proportions. There is no possibility of substituting labor for capital in production. If this assumption is abandoned, the knife-edge notion of unstable balance seems to go with it." (Solow 1956, 65) **As a result, possibility of substituting labor for capital in production, so, the adaptation between warranted rate of growth and natural rate of growth cannot be explained except for out-of equilibrium conditions.**

The neoclassical adaptation mechanism depends on the factor markets: "In general if a stable growth path exists, the fall in the real wage or real rental needed to get to it may not be catastrophic at all. If there is an initial shortage of labor (compared with the equilibrium ratio) the real wage will have to fall. The higher the rate of increase of the labor force and the lower the propensity to save, the lower the equilibrium ratio and hence the more the real wage will have to fall." (Solow 1956, 73) Hence, of course, **if out-of equilibrium conditions cannot be occurred, then, transition from out-of-equilibrium to full employment conditions, so, the neoclassical adaptation mechanism based on factor markets cannot be explained either.**

Thus, in the presence of steady-state equilibrium, the original Solow model explains the stable relation between the natural and the warranted rates of growth based on the idea that relative factors proportions are monotonically and inversely related to the ratio of factor returns, i.e. the neoclassical adaptation mechanism. This adaptation mechanism works through factor markets. If the natural rate of growth differs from the warranted rate, there is a mismatch between the supply of and demand for labor and capital. Thus, real wages and rental cost of capital change and the equilibrium between (a) the supply of and demand for labor, and capital (b) the natural rate and the warranted rates of growth are sustained. In other words, the warranted rate adapts itself to the natural rate based on the idea that relative factors proportions are monotonically and inversely related to the ratio of factor returns.

However, if there are stationary-state conditions, output growth will equal to zero, saving propensity cannot change exogenously, economy cannot move away from equilibrium. This result emphasizes that if it is assumed to be stationary-state conditions, the neoclassical postulate which is explained above will not be valid. On the other hand, if there are steady-state conditions, output growth is positive, saving propensity can change exogenously and the economy may move away from equilibrium.

A numerical example

We now provide a numerical example to explain stable equilibrium assuming steady-state and stationary-state, respectively.

Assume an economy where $n = 0.03$, $K_0 = 10{,}000$, $\delta = 0$ $Y_0 =$ 2,000 and $s_0 =$ 0.15. Assume that the growth rate of technology is zero. Then at the initial period (t = 0), the following equations:

$nK_0 = 0.03 \text{ x } 10{,}000 = 300$

$s_0 Y_0 = 0.15 \text{ x } 2{,}000 = 300.$

Since there are constant returns to scale and the economy remains on a balanced growth path, the capital stock grows at a rate equal to the growth rate of the labor force, which is 0.03. In addition, assume that the elasticity of the output is 0.7 and 0.3 for capital and labor, respectively. Then the growth rate of output is as follows[4]:

$$\frac{\Delta Y_1}{Y_0} = 0.7 \text{ x } 0.03 + 0.3 \text{ x } 0.03 = 0.03.$$

Then at the next period (t = 1) the levels of output and capital stock are as follows:

$$Y_1 = Y_0 + \frac{\Delta Y_1}{Y_0} \text{ x } Y_0 = 2{,}000 + 0.03 \text{ x } 2{,}000 = 2{,}060$$

$$K_1 = K_0 + \frac{\Delta K_1}{K_0} \text{ x } K_0 = 10{,}000 + 0.03 \text{ x } 10{,}000 = 10{,}300.$$

Note that saving rate does not change ($s_0 = s_1 =$ 0.10.). At t = 1, the fundamental equation is

$s_1 Y_1 = nK_1$

0.15 x 2,060 = 0.03 x 10,300

309 = 309.

Now assume that at the next period (t = 2), saving propensity exogenously rises and is equal to 0.20. **This can be occurred if there is some positive yield of capital stock.** For period 2, before a change occurs in output, investment is equal to

$s_2 Y_1 = 0.20 \text{ x } 2{,}060 = 412.$

Then, in period 2, before a change occurs in output, nK_1 is less than $s_2 Y_1$:

$nK_1 = 309 < s_2 Y_1 = 412$

$s_2 Y_1 - nK_1 = 412 - 309 = 103$

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[4] Under constant to returns to scale assumption, production function is as follows: $Y_t = K_t^{\alpha} L_t^{1-\alpha}$ where α and $1-\alpha$ are the elasticity of the output (Y) for capital (K) and labor (L), respectively. This production function is written using growth rates as follows: $\frac{\Delta Y_t}{Y_{t-1}} = \alpha \, x \frac{\Delta K_t}{K_{t-1}} + (1-\alpha) x \frac{\Delta L_t}{L_{t-1}}$.

Since $s_2 Y_1 = \Delta K_2 = 412$, the capital stock at period 2 is given by

$$K_2 = K_1 + \Delta K_2 = 10{,}300 + 412 = 10{,}712.$$

Then, the growth rate of the capital stock and output will be

$$\frac{\Delta K_2}{K_1} = \frac{412}{10{,}300} = 0.04$$

$$\frac{\Delta Y_2}{Y_1} = 0.7 \text{ x } 0.04 + 0.3 \text{ x } 0.03 = 0.037.$$

It can be seen that if saving propensity rises, the capital stock grows faster than the growth rate of the labor force, the required saving is less than the desired investment, so economy moves away from equilibrium. Now, since the growth rate of the capital stock is greater than the growth rate of the labor force, due to the diminishing returns, the economy moves towards to the steady-state capital-to-labor ratio.

Thus, this example clarifies why an economy moves away from equilibrium: it does so because of an exogenous change in saving propensity[5]. This change results a difference between the warranted and the natural rates of growth, i.e. between the growth rates of capital stock and the labor force, respectively.

Let us to provide this numerical example to explain stable equilibrium based on textbook explanation. Assume an economy at the stationary state; i.e. $n = 0.00$. Besides, $K_0 = 10{,}000$, $Y_0 = 2{,}000$, $\delta = 0.03$ and it is assumed that the growth rate of technology is zero. Since, it is assumed that economy is at the stationary state there are no net savings, and so net investment. On the other hand gross investment is equal to the depreciation. Thus, assuming that $s = 0.15$, at the initial period (t = 0), the following equations:

$$\delta K_0 = 0.03 \text{ x } 10{,}000 = 300$$

$$s_0 Y_0 = 0.15 \text{ x } 2{,}000 = 300.$$

Since there are constant returns to scale and the economy remains on a balanced growth path, the capital stock grows at a rate equal to the growth rate of the labor force, which is 0.00. Remember that the assumption that the elasticity of the output is 0.7 and 0.3 for capital and labor, respectively. Then the growth rate of output is as follows:

$$\frac{\Delta Y_1}{Y_0} = 0.7 \text{ x } 0.00 + 0.3 \text{ x } 0.00 = 0.00.$$

Then at the next period (t = 1) the levels of output and capital stock are as follows:

[5] Note that, out-of-equilibrium conditions can emerge, in the absence of technical change, due to changes in either the saving propensity, the growth rate of the labor force or the depreciation rate. However, as noted at the footnote 3, the Solow model assumes growth rate of the labor force and the depreciation rate exogenous and the saving propensity should be focused as a factor that causes the economy to move away from equilibrium because it can change due to the decisions of the agents.

$$Y_1 = Y_0 + \frac{\Delta Y_1}{Y_0} \text{ x } Y_0 = 2{,}000 + 0.00 \text{ x } 2{,}000 = 2{,}000$$

$$K_1 = K_0 + \frac{\Delta K_1}{K_0} \text{ x } K_0 = 10{,}000 + 0.00 \text{ x } 15{,}000 = 10{,}000.$$

At t = 1, the fundamental equation is

$$s_1 Y_1 = \delta K_1$$

0.15 x 2,000 = 0.03 x 10,000

300 = 300

Now assume that at the next period (t = 2), saving propensity exogenously increases. However, it cannot be possible since yield of capital is zero; i.e. $\Delta K_1 = 0$, and so $\Delta Y_1 = 0$.

It must be emphasized that if there are stationary-state, rather than steady-state conditions, there is no economic growth and there are no factors pushing the economy to move away from equilibrium; i.e. saving propensity cannot change at the stationary-state. However, if there are steady-state, rather than stationary-state conditions, there is a positive economic growth, saving propensity can change and the economy may move away from equilibrium.

Thus, this example clarifies that textbook explanation based on simplification of the Solow model cannot explain why an economy moves away from equilibrium; i.e. saving propensity cannot change at the stationary-state.

An offer to explain stability in the Solow model

First, accumulation of capital is given by

$$\Delta k(t) = sy(t) - (n + \delta)k(t) \tag{5}$$

where $k(t)$ is the per capita capital stock, s is the saving rate, $y(t)$ is the per capita output, n is the growth rate of the labor force and δ is the depreciation rate. Given that steady-state conditions growth rate of labor and capital stock equal to n and we rewrite (5) as follows:

$$sy(t) = (n + \delta)k(t) \tag{6}$$

Assume that growth rate of labor force and the depreciation rate are constant, and because of a saving propensity above its steady-state value, the economy has a capital-to-labor ratio below the steady-state value. Thus the desired saving is greater than the required investment:

$$sy(t) > (n + \delta)k(t) \tag{7}$$

Then excess saving is invested, capital accumulates and the economy moves to the steady-state value of $k(t)$. This means that the warranted rate of growth (the growth rate of the capital) is greater than the natural rate of growth (the growth rate of the labor force).

Note that, this process occurs because rational producers choose more capital-intensive production techniques since real user cost of capital is lower than real wage. Thus, as more capital-intensive production techniques is preferred to labor-intensive production techniques, capital is substituted to labor until to the steady state.

Besides, this process continues until the steady-state value of $k(t)$ is reached, at a diminishing rate of growth thanks to the diminishing returns to capital. At the steady -state value of $k(t)$, growth rate of capital stock is equal to labor and $k(t)$ remains constant; i.e. the warranted rate of growth (the growth rate of the capital) is equal to the natural rate of growth (the growth rate of the labor force).

Conclusion

In the foregoing, it was shown that when the Solow model is explained under the assumption that the growth rate of the labor force is equal to zero, saving propensity cannot change exogenously. The validity of the assumption is called into question by the fact that it cannot give an explanation for either the neoclassical adaptation mechanism or the relation between the natural and the warranted rates of growth. In contrast, faithfulness to Solow's original text when explaining the Solow model provides a consistent approach to its neoclassical background.

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